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CP-violating Majorana phases, lepton-conserving processes and final state interactions

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Abstract

The CP-violating phases associated with Majorana neutrinos can give rise to CP-violating effects even in processes that conserve total Lepton number, such as $\mu \to ee\bar{e}$, $\mu e \to ee$ and others. After explaining the reasons that make this happen, we consider the calculation of the rates for the process of the form $\ell_a\ell_b \to \ell_a\ell_c$ and its conjugate $\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c$, where ℓ_a, ℓ_b, ℓ_c denote charged leptons of different flavors. In the context of the Standard Model with Majorana neutrinos, we show that the difference in the rates depends on such phases. Our calculations illustrate in detail the mechanics that operate behind the scene, and set the stage for carrying out the analogous, more complicated (as we explain), calculations for other processes such as $\mu \to ee\bar{e}$ and its conjugate.

1 Introduction

Sometime ago [1] we introduced a prescription for identifying a minimal set of CP-violating parameters in the lepton sector that are invariant under a rephasing of the fermion fields. The prescription holds for any number of fermion generations and, more importantly, accommodate the case in which the neutrinos are Majorana particles. Recently [2], based on that work we analyzed the dependence of the squared amplitudes on the rephasing-invariants of the lepton sector with Majorana neutrinos, for various lepton-violating or lepton-conserving processes, giving special attention to the dependence on the extra CP-violating parameters that are due to the Majorana nature of the neutrinos [3, 4, 5].

It was widely believed that these extra parameters appear only in lepton number violating processes [6], and there is a lot of discussion in the literature about their possible observable effects[7, 8, 9, 10, 11, 12]. However, in our recent work [2], we showed that they can appear in lepton-number conserving processes as well. The true condition for the occurrence of these parameters in a given process seems to be the violation of lepton number on any fermion line in the corresponding diagrams, and not necessarily that total lepton number be violated by the process as a whole. In processes that conserve the total lepton number, there are in general diagrams in which the individual fermion lines change the lepton number, but do so in such a way that the changes between different lines cancel in the overall diagram. The interference terms produced by such diagrams contain the extra CP-violating parameters that exist due to the Majorana nature of the neutrinos.

However, this analysis was carried out by considering various generic physical processes, classified according to whether they conserve total lepton number, or by how many units they violate it, and then finding their generic dependence on the rephasing-invariant CP-violating parameters. The question of what is the mechanics that operates in a specific process to give rise to such effects was not considered there. This question is important when we attempt to consider the difference in the rates for, for example, $\mu \to ee\bar{e}$ vs $\bar{\mu} \to \bar{e}\bar{e}e$, due to the CP violating phases. The issue here is that, unless the final-state interactions are taken into account, the calculation of the two rates will be equal (by the CPT theorem) in spite of the fact that CP may be violated. Thus, while the arguments and analysis of Ref. [2] are indicative, a specific calculation of the effects depends in general on the kinematical and dynamical aspects of the particular process considered.

In order to fill this gap, we consider in the present paper the calculation of the difference of the rates for some leptonic processes and their conjugate ones, in the context of the Standard Model with Majorana neutrinos. The processes that we consider have the virtue that they have a two-body final state, which makes it simpler to take into account the final-state interactions, yet they contain all the ingredients to understand and illustrate the issues that we have mentioned. In addition, some of the formulas that we will present on the way, are also required ingredients in the corresponding calculations for other processes such as $\mu \to ee\bar{e}$.

In Section 2 we make some general remarks about the type of process we consider, explain why the effect can be seen in some of the processes and not in others, mention the need to consider the effect of the final state interactions, and set the stage for the calculations that make up the rest of the paper. The effect of the final state interactions, which show up as an absorptive term in the total amplitude, is calculated in Section 3, for both the direct process and its conjugate. Based on those results, in Section 4 we compute the difference of the differential rates for the process and its conjugate, and thus we are able to show explicitly that it is given in terms of the rephasing-invariant CP-violating parameters for Majorana neutrinos. Our outlook and conclusions are given in Section 5. Four appendices contain some of the details of various stages of the calculation, including a Fierz transformation formula used, the Cutkosky rules employed to determine the absorptive term due to the final state interactions, and the phase space integrals over the intermediate states required to implement the Cutkosky formula.

2 General Remarks

We want to consider processes of the generic form

$$\ell_a \ell_b \to \ell_c \ell_d \tag{2.1}$$

where a, b, c, d take values from anyone of the lepton flavors e, μ, τ, with the condition that they are not all equal. Such processes conserve total lepton number but in general violate the individual lepton flavors. By the usual substitution (crossing) rules, our considerations also apply to those processes that are related to these by crossing.

We classify the processes into two groups, depending on whether or not the diagrams that contain the one-loop photon or Z vertex functions also contribute. We denote by Group I the set of processes for which the only diagrams that contribute are the box diagrams, as shown in Fig. 1. The requirement for the photon and Z vertex diagrams to be absent is that

$$\ell_c \neq \ell_a \quad \text{and} \quad \ell_c \neq \ell_b \tag{2.2}$$

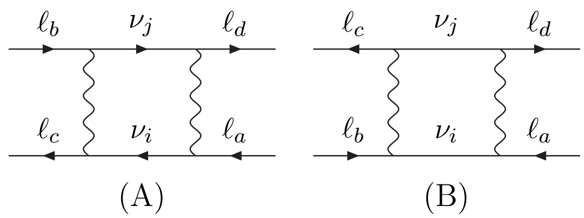


Figure 1: The physical diagrams for the general process $\ell_a \ell_b \to \ell_c \ell_d$. The unlabeled vector boson lines represent the W vector boson. In addition to these diagrams, there are other diagrams in which any one of the W bosons are replaced by their corresponding unphysical Higgs particle, plus the exchange diagrams in which ℓ_c and ℓ_d interchanged. Note that diagram (B) contributes only if the neutrinos are Majorana particles.

and

$$\ell_d \neq \ell_a \quad \text{and} \quad \ell_d \neq \ell_b \tag{2.3}$$

Because each of the four indices a, b, c, d can only take three possible values (e, μ, τ), it is clear that the conditions in (2.2) and (2.3) can be satisfied simultaneously only if either

$$\ell_a = \ell_b \,, \tag{2.4}$$

or if

$$\ell_c = \ell_d \,. \tag{2.5}$$

Without any loss in generality, we can fix one or the other of the two possibilities, and thus define the processes in Group I as those of the general form

$$\ell_a \ell_b \leftrightarrow \ell_c \ell_c \quad (\ell_a \neq \ell_b \neq \ell_c) \,. \tag{2.6}$$

On the other hand, the processes in Group II are those for which one (or more) of the conditions in (2.2) and (2.3) is not satisfied. Again, without loss of generality, they can be represented by the general form of Eq. (2.1) with $\ell_d = \ell_a$, i.e., these processes have the generic form

$$\ell_a \ell_b \to \ell_a \ell_c \,, \tag{2.7}$$

with $c \neq b$ being the only restriction. In this case, in addition to the diagrams shown in Fig. 1, the diagrams shown schematically in Fig. 2 must be included.

2.1 No-go result for Group I

As already stated, the diagrams that contribute to the amplitude for these processes are the box diagrams shown in Fig. 1. To the leading order $(1/M_W^4)$, only the diagrams involving the W exchange are important. Let us consider $\ell_a \ell_b \to \ell_c \ell_c$. The physical amplitude is of the form

$$\begin{aligned} A\Big(\ell_a(k)\ell_b(p) \to \ell_c(p_1)\ell_c(p_2)\Big) &= M_A\Big(\ell_a(k)\ell_b(p) \to \ell_c(p_1)\ell_c(p_2)\Big) \\ &\quad + M_B\Big(\ell_a(k)\ell_b(p) \to \ell_c(p_1)\ell_c(p_2)\Big) \end{aligned} \tag{2.8}$$

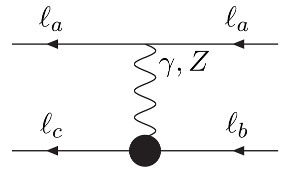


Figure 2: Representation of the collection of diagrams for the processes in Group II, that involve the one-loop $\gamma\ell_b\ell_c$ or $Z\ell_b\ell_c$ vertex functions.

where $M_X(\ell_a(k)\ell_b(p) \to \ell_c(p_1)\ell_c(p_2))$ is the contribution from each diagram. Their calculation is straightforward, and some of the details are provided in Appendix A. The results for the present case are summarized by the formulas

$$\begin{aligned} M_A\Big(\ell_a(k)\ell_b(p) \to \ell_c(p_1)\ell_c(p_2)\Big) &= \lambda_A^{(abcc)}[\bar{u}_c(p_2)\gamma^\mu L u_b(p)][\bar{u}_c(p_1)\gamma_\mu L u_a(k)]\,, \\ M_B\Big(\ell_a(k)\ell_b(p) \to \ell_c(p_1)\ell_c(p_2)\Big) &= \lambda_B^{(abcc)}[\bar{u}_c(p_2)\gamma^\mu L u_b(p)][\bar{u}_c(p_1)\gamma_\mu L u_a(k)]\,, \end{aligned} \tag{2.9}$$

where

$$\begin{aligned} \lambda_A^{(abcd)} &\equiv \frac{g^4}{64\pi^2 M_W^2}\sum_{i,j}(V^*_{ai}V^*_{bj}V_{ci}V_{dj})(4f_A^{(ij)}) + (c \leftrightarrow d)\,, \\ \lambda_B^{(abcd)} &\equiv \frac{g^4}{64\pi^2 M_W^2}\sum_{i,j}\Big(K_i^2 K_j^{*\,2} V^*_{ai}V^*_{bi}V_{cj}V_{dj}\Big)(2f_B^{(ij)}) + (c \leftrightarrow d)\,. \end{aligned} \tag{2.10}$$

The functions $f_A^{(ij)}$ and $f_B^{(ij)}$ appearing in these equations are given by

$$\begin{aligned} f_A^{(ij)} &= \frac{r_i^2\log r_i}{r_j - r_i} - r_i + (i \leftrightarrow j)\,, \\ f_B^{(ij)} &= \frac{m_{\nu_i}m_{\nu_j}}{M_W^2}\left\{\frac{r_i\log r_i}{r_j - r_i} - r_i + (i \leftrightarrow j)\right\}\,, \end{aligned} \tag{2.11}$$

where

$$r_i = \frac{m_{\nu_i}^2}{M_W^2}\,. \tag{2.12}$$

Further, the V_{ai} are the elements of the lepton mixing matrix, and the K_i are the phases defined by the Majorana condition

$$\nu_i^c = K_i^2\nu_i\,. \tag{2.13}$$

Although there are several equivalent ways to write the results for the amplitudes $M_{A,B}$, they can be brought to this form by suitable Fierz transformations.

The upshot of this is that when Eq. (2.9) is substituted in Eq. (2.8), the effective couplings $\lambda_{A,B}$ appear in combination as a common overall factor of the total amplitude. Since there is no interference term, the rate for the process and its conjugate is the same, and there is no observable CP violating effect in this type of process. We have considered explicitly the amplitude for $\ell_a\ell_b \to \ell_c\ell_c$, but similar arguments hold for the inverse $\ell_c\ell_c \to \ell_a\ell_b$, whose amplitude is simply the complex conjugate, and other related processes such as $\ell_a \to \bar{\ell}_b\ell_c\ell_c$.

While we have summarized the result of the actual calculation, a little thought reveals what is going on. Because we are calculating to leading order in $1/M_W^2$, the dominant terms come from the W-exchange diagrams, as we have already mentioned. The chiral nature of the W interactions dictate that, to leading order, only the left-handed components of the external fermion fields enter in the amplitude. The most economical way to express this fact is by writing down the effective Lagrangian for this process which, by the above argument, can only be of the form

$$\mathscr{L}^{(W)} = \frac{\lambda}{2}[\bar{\ell}_c\gamma^\mu L\ell_b][\bar{\ell}_c\gamma_\mu L\ell_a] + \text{h.c.} \tag{2.14}$$

In fact, the results given in Eqs. (2.8) and (2.9) can be represented by this Lagrangian, with the identification $\lambda = \lambda_A^{(abcc)} + \lambda_B^{(abcc)}$. Thus, to this order the effective Lagrangian actually consists of only one term, and therefore the rates for the process and its conjugate are equal.

2.2 Evasion for Group II

By the same argument, it is now easy to see how the processes in Group II differ. We consider specifically those with $\ell_a \neq \ell_b$. The diagrams for a process of the generic form given in Eq. (2.7), include the diagrams that involve the $\gamma\ell_b\ell_c$ and $Z\ell_b\ell_c$ one-loop vertex functions [13]. Instead of Eq. (2.8), the physical amplitude in this case is of the form

$$\begin{aligned} A\Big(\ell_a(k)\ell_b(p) \to \ell_a(k')\ell_c(p')\Big) = {} & (\lambda_A^{(abac)} + \lambda_B^{(abac)})[\bar{u}_a(k')\gamma^\mu L u_a(k)][\bar{u}_c(p')\gamma_\mu L u_b(p)] \\ & + \lambda_Z^{(bc)}[\bar{u}_a(k')\gamma^\mu(X + Y\gamma_5)u_a(k)][\bar{u}_c(p')\gamma_\mu L u_b(p)] \,, \end{aligned} \tag{2.15}$$

where

$$\lambda_Z^{(bc)} = -\frac{g^4}{64\pi^2 M_W^2}\sum_k V_{bk}^* V_{ck} f_Z^{(k)} \,, \tag{2.16}$$

with

$$f_Z^{(k)} = r_k \log r_k \,, \tag{2.17}$$

while X and Y are the neutral-current couplings of the lepton ℓ_a,

$$\begin{aligned} X &= -\frac{1}{2} + \sin^2\theta_W \\ Y &= \frac{1}{2} \,. \end{aligned} \tag{2.18}$$

We mention the following. In the formula quoted in Eq. (2.16) we have neglected the other terms of order $1/M_W^4$ that do not contain the logarithmic factor $\log r_i$. In addition, none of the terms that arise from the diagram that involve the photon vertex function contain that logarithmic factor, and therefore we have omitted altogether that contribution in Eq. (2.15). Thus, by the same argument that led us to write Eq. (2.14), this amplitude corresponds to an effective Lagrangian

$$\begin{aligned} \mathscr{L}^{(W+Z)} = {} & (\lambda_A^{(abac)} + \lambda_B^{(abac)})[\bar{\ell}_a\gamma^\mu L\ell_a][\bar{\ell}_c\gamma_\mu L\ell_b] \\ & + \lambda_Z^{(bc)}[\bar{\ell}_a\gamma^\mu(X + Y\gamma_5)\ell_a][\bar{\ell}_c\gamma_\mu L\ell_b] + \text{h.c.}\,. \end{aligned} \tag{2.19}$$

This in turn can be written in the more compact form

$$\mathscr{L}^{(W+Z)} = [\bar{\ell}_a \gamma^\mu (X' + Y'\gamma_5)\ell_a][\bar{\ell}_c \gamma_\mu L \ell_b] + h.c. \,, \tag{2.20}$$

with

$$\begin{aligned} X' &= \lambda_A^{(abac)} + \lambda_B^{(abac)} + \lambda_Z^{(bc)} X \\ Y' &= -(\lambda_A^{(abac)} + \lambda_B^{(abac)}) + \lambda_Z^{(bc)} Y \,. \end{aligned} \tag{2.21}$$

In particular, X' and Y' are complex quantities, and depend on the Majorana phases through the λ_B term.

A kinematic observable that depends on the interference term $\mathrm{Im}\,(X'^* Y')$ will be sensitive to the CP-violating Majorana phases. However, the total rate, determined from this effective Lagrangian, or equivalently calculated with the amplitude given in Eq. (2.15), will not depend on that interference term. The reason is that, as seen from Eq. (2.20), such term arises from the interference between the vector and axial vector parts of the current, and that vanishes after summing and averaging over the polarizations and integrating over the phase space. Moreover, the rates for any process and its conjugate, determined from Eq. (2.20), are equal. This is ultimately due to the fact that, while CP does not hold in Eq. (2.20), CPT does hold and that is sufficient to guarantee the equality of the rates.

When we consider the final state interactions between the outgoing leptons, this is no longer true as is well known. While the amplitude for the direct process depends on X' and Y', the amplitude for the conjugate process depends on the complex conjugates of these two quantities. However, the final state interactions induce an extra phase that is the same for both the direct process and the conjugate. This mismatch between the two sets of phases in both cases leads to the inequality of the total rates. In the language of the effective Lagrangian, the effect of the final state interactions is to augment Eq. (2.20) in a way that renders it non-hermitian. Our task in the next section is to calculate the effect that we have just outlined.

We would like to remark that the above statements about the equality or inequality of the total rates, as the case may be, apply to the total differential rates as well. The latter quantities are defined from the squared amplitude as usual, by summing over the final spins and averaging over the initial ones, with the total integrated rates being obtained from them by carrying out the only non-trivial integration, over the azimuthal angle. The reason why we can also consider the total differential rates is the following. Under a CP or CPT transformation, the amplitude is related to the amplitude for the conjugate process, with perhaps the momentum and/or spin variables reversed. For the total integrated rates, the reversal of the momentum and spin variables is of no consequence since they are being summed over. In the case of the total differential rates, the reversal of the spin variables is not relevant either for the same reason. On the other hand, for the two body processes that we are considering, the momentum vectors can appear only through the scalar variables formed out of the scalar products among them, and those variables are unchanged by the simultaneous transformations of the momentum vectors[1].

[1]The situation is different if we consider for example, processes with three particles in the final state. In that case, the spin-averaged squared amplitude can contain momentum contractions involving the four-dimensional antisymmetric tensor, and those change when the momentum vectors are reversed. Our considerations could be applied to such cases also, but only if some (non-trivial) angular integrations are made so that those terms do not appear in some specially designed differential rates.

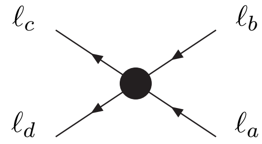
$iM^{(0)} =$ 

Figure 3: Schematic representation of the one-loop amplitude for $\ell_a \ell_b \to \ell_c \ell_d$. The diagram stands for the collection of diagrams referred to in Figs. 1 and 2.

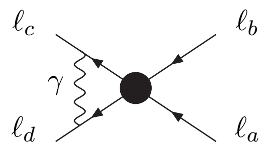
$iM^{(1)} =$ 

Figure 4: Schematic representation of the correction to include the electromagnetic final state interactions in $\ell_a \ell_b \to \ell_c \ell_d$.

3 Final state interactions

3.1 General considerations

We denote by $M^{(0)}(\ell_a \ell_b \to \ell_a \ell_c)$ the amplitude for $\ell_a \ell_b \to \ell_a \ell_c$ determined from Eq. (2.20), which we represent schematically in Fig. 3. Analogously, we denote by $M^{(0)}(\bar{\ell}_a \bar{\ell}_b \to \bar{\ell}_a \bar{\ell}_c)$ the amplitude for the conjugate process. As we have already mentioned, since the individual terms that contribute to $M^{(0)}$ do not contain an absorptive part, then the rates $\Gamma(\ell_a \ell_b \to \ell_a \ell_c)$ and $\Gamma(\bar{\ell}_a \bar{\ell}_b \to \bar{\ell}_a \bar{\ell}_c)$ calculated with the above amplitudes are equal, as a consequence of CPT. This result does not hold if we include the final state interactions between the two leptons, of which the dominant one is the electromagnetic interaction. Diagrammatically, the additional terms are represented in Fig. 4. The total amplitude is

$$M = M^{(0)} + M^{(1)} \,. \tag{3.1}$$

Using the Cutkosky rules, the absorptive part of $M^{(1)}$ can be expressed in terms of essentially $M^{(0)}$ itself, times some factors.

In order to write the following formulas in a compact form, let us introduce the following notation. We denote the initial and the final states in the process $\ell_a \ell_b \to \ell_a \ell_c$ by

$$\begin{aligned} |i\rangle &\equiv |\ell_a(k, s_a)\ell_b(p, s_b)\rangle \\ |f\rangle &\equiv |\ell_a(k', s'_a)\ell_c(p', s'_c)\rangle \,. \end{aligned} \tag{3.2}$$

The S-matrix element for this process is then written as

$$\langle f\,|S|\,i\rangle = (2\pi)^4 \delta^{(4)}(k + p - k' - p')\Big[iM^{(0)}(\ell_a \ell_b \to \ell_a \ell_c) + iM^{(1)}(\ell_a \ell_b \to \ell_a \ell_c)\Big] \,, \tag{3.3}$$

where $M^{(0)}(\ell_a \ell_b \to \ell_a \ell_c)$ is determined from Eq. (2.20). Our next task is to find the appropriate expression for the two-loop amplitude $M^{(1)}(\ell_a \ell_b \to \ell_a \ell_c)$.

In general, it can be decomposed as

$$M^{(1)}(\ell_a\ell_b \to \ell_a\ell_c) = M^{(1)}_{\rm disp}(\ell_a\ell_b \to \ell_a\ell_c) + M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c)\,, \tag{3.4}$$

where $M^{(1)}_{\rm disp}$ and $M^{(1)}_{\rm abs}$ stand for its dispersive and absorptive parts, respectively. Since, as already mentioned, $M^{(0)}$ does not contain an absorptive part, the contribution from $M^{(1)}_{\rm disp}$ only produces an order α correction to $M^{(0)}$ which we neglect. On the other hand, although $M^{(1)}_{\rm abs}$ is also a factor of order α smaller than $M^{(0)}$, it is the piece we are after since it contributes an absorptive part to the full amplitude. To calculate it, we employ the Cutkosky rules [14]. As shown in Appendix C, in our notation for the present case they yield

$$M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c) = \frac{i}{2}\sum_n (2\pi)^4\delta^{(4)}(q_1+q_2-k'-p')M^{(\gamma)}\Big(\ell_a^{(n)}\ell_c^{(n)} \to \ell_a\ell_c\Big)M^{(0)}\Big(\ell_a\ell_b \to \ell_a^{(n)}\ell_c^{(n)}\Big)\,. \tag{3.5}$$

Here the symbols $\ell_a^{(n)}\ell_c^{(n)}$ stand for an intermediate $\ell_a\ell_c$ state

$$|n\rangle \equiv |\ell_a(q_1,\sigma_1)\ell_c(q_2,\sigma_2)\rangle\ , \tag{3.6}$$

and the sum over the intermediate states stands for

$$\sum_n \to \int \frac{d^3q_1}{(2\pi)^3 2E_1^{(a)}}\frac{d^3q_2}{(2\pi)^3 2E_2^{(c)}}\sum_{\sigma_1,\sigma_2}\ . \tag{3.7}$$

The quantity $M^{(\gamma)}$ in Eq. (3.5) is the electromagnetic scattering amplitude for $\ell_a(q_1,\sigma_1) + \ell_c(q_2,\sigma_2) \to \ell_a(k',s'_a) + \ell_c(p',s'_c)$, i.e.,

$$iM^{(\gamma)}\Big(\ell_a^{(n)}\ell_c^{(n)} \to \ell_a\ell_c\Big) = \frac{ie^2}{(q_1-k')^2}\left[\bar u_a(k',s'_a)\gamma^\mu u_a(q_1,\sigma_1)\right]\left[\bar u_c(p',s'_c)\gamma_\mu u_c(q_2,\sigma_2)\right]\ . \tag{3.8}$$

Thus finally, the full amplitude is given by

$$M(\ell_a\ell_b \to \ell_a\ell_c) = M^{(0)}(\ell_a\ell_b \to \ell_a\ell_c) + M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c)\,, \tag{3.9}$$

where $M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c)$ is computed from Eq. (3.5), with $M^{(0)}(\ell_a\ell_b \to \ell_a\ell_c)$ determined from (2.20).

In a similar fashion, for the conjugate process $\bar\ell_a\bar\ell_b \to \bar\ell_a\bar\ell_c$, we define

$$\begin{aligned} |\bar i\rangle &\equiv |\bar\ell_a(k,s_a)\bar\ell_b(p,s_b)\rangle \\ |\bar f\rangle &\equiv |\bar\ell_a(k',s'_a)\bar\ell_c(p',s'_c)\rangle \\ |\bar n\rangle &\equiv |\bar\ell_a(q_1,\sigma_1)\bar\ell_c(q_2,\sigma_2)\rangle\ . \end{aligned} \tag{3.10}$$

Neglecting again the dispersive part of the amplitude, the corresponding S-matrix element is given by

$$\langle\bar f\,|S|\,\bar i\rangle = (2\pi)^4\delta^{(4)}(k+p-k'-p')\Big[iM^{(0)}(\bar\ell_a\bar\ell_b \to \bar\ell_a\bar\ell_c) + iM^{(1)}_{\rm abs}(\bar\ell_a\bar\ell_b \to \bar\ell_a\bar\ell_c)\Big]\,. \tag{3.11}$$

where

$$M^{(1)}_{\rm abs}(\bar\ell_a\bar\ell_b \to \bar\ell_a\bar\ell_c) = \frac{i}{2}\sum_{\bar n}(2\pi)^4\delta^{(4)}(q_1+q_2-k'-p')M^{(\gamma)}\Big(\bar\ell_a^{(n)}\bar\ell_c^{(n)} \to \bar\ell_a\bar\ell_c\Big)M^{(0)}\Big(\bar\ell_a\bar\ell_b \to \bar\ell_a^{(n)}\bar\ell_c^{(n)}\Big)\,, \tag{3.12}$$

with

$$iM^{(\gamma)}\left(\bar{\ell}_a^{(n)}\bar{\ell}_c^{(n)} \to \bar{\ell}_a\bar{\ell}_c\right) = \frac{ie^2}{(q_1 - k')^2}\left[\bar{v}_a(q_1,\sigma_1)\gamma^\mu v_a(k',s'_a)\right]\left[\bar{v}_c(q_2,\sigma_2)\gamma_\mu v_c(p',s'_c)\right] . \quad (3.13)$$

It is useful to note the following. Using the relation between the spinors $u(p,s)$ and $v(p,s)$, e.g.,

$$v(p,s) = i\gamma_2 u^*(p,s)\,, \quad (3.14)$$

in a specific convention, together with relations such as

$$\bar{v}'\gamma_\mu v = \bar{u}\gamma_\mu u'\,, \quad (3.15)$$

it follows that

$$M^{(\gamma)}\left(\bar{\ell}_a^{(n)}\bar{\ell}_c^{(n)} \to \bar{\ell}_a\bar{\ell}_c\right) = M^{(\gamma)}\left(\ell_a^{(n)}\ell_c^{(n)} \to \ell_a\ell_c\right), \quad (3.16)$$

that is, the electromagnetic amplitude for the two processes is the same.

Our task at hand is to apply these formulas to compute the absorptive part of the amplitudes for the direct process and its conjugate, from Eqs. (3.5) and (3.12), respectively.

3.2 The absorptive part $M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c)$

First of all, from Eq. (2.20) we write

$$M^{(0)}\left(\ell_a\ell_b \to \ell_a^{(n)}\ell_c^{(n)}\right) = [\bar{u}_a(q_1)\gamma^\mu(X' + Y'\gamma_5)u_a(k)][\bar{u}_c(q_2)\gamma_\mu L u_b(p)]\,. \quad (3.17)$$

Using this and Eq. (3.8), we then obtain

$$\begin{aligned} M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c) = {} & \frac{ie^2}{2}\int dL[\bar{u}_a(k')\gamma^\lambda(\not{q}_1 + m_a)\gamma^\mu(X' + Y'\gamma_5)u_a(k)] \\ & \times[\bar{u}_c(p')\gamma_\lambda(\not{q}_2 + m_c)\gamma_\mu L u_b(p)]\,, \end{aligned} \quad (3.18)$$

where the symbol $\int dL$ stands for

$$\int dL \to \int \frac{d^3q_1}{(2\pi)^3 2E^{(a)}_{q_1}}\frac{d^3q_2}{(2\pi)^3 2E^{(c)}_{q_2}}(2\pi)^4\delta^{(4)}(k' + p' - q_1 - q_2)\frac{1}{(k' - q_1)^2}\,. \quad (3.19)$$

In order to proceed, we introduce the following definitions for the integrals over the intermediate momenta

$$\begin{aligned} I^{(0)} &= \int dL \\ I^{(1)}_\mu &= \int dL\; q_{1\mu} \\ I^{(2)}_\mu &= \int dL\; q_{2\mu} \\ I^{(12)}_{\mu\nu} &= \int dL\; q_{1\mu}q_{2\nu}\,, \end{aligned} \quad (3.20)$$

in terms of which

$$M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c) = \frac{ie^2}{2}\left(M^{(1)}_1 + M^{(1)}_2 + M^{(1)}_3 + M^{(1)}_4\right), \quad (3.21)$$

where

$$
\begin{aligned}
M_1^{(1)} &= m_a m_c I^{(0)} \Big[\bar{u}_a(k')\gamma^\lambda\gamma^\mu(X' + Y'\gamma_5)u_a(k)\Big] \Big[\bar{u}_c(p')\gamma_\lambda\gamma_\mu L u_b(p)\Big] \\
M_2^{(1)} &= m_a I^{(2)}_\rho \Big[\bar{u}_a(k')\gamma^\lambda\gamma^\mu(X' + Y'\gamma_5)u_a(k)\Big] \Big[\bar{u}_c(p')\gamma_\lambda\gamma^\rho\gamma_\mu L u_b(p)\Big] \\
M_3^{(1)} &= m_c I^{(1)}_\rho \Big[\bar{u}_a(k')\gamma^\lambda\gamma^\rho\gamma^\mu(X' + Y'\gamma_5)u_a(k)\Big] \Big[\bar{u}_c(p')\gamma_\lambda\gamma_\mu L u_b(p)\Big] \\
M_4^{(1)} &= I^{(12)}_{\rho\nu} \Big[\bar{u}_a(k')\gamma^\lambda\gamma^\rho\gamma^\mu(X' + Y'\gamma_5)u_a(k)\Big] \Big[\bar{u}_c(p')\gamma_\lambda\gamma^\nu\gamma_\mu L u_b(p)\Big] .
\end{aligned}
\tag{3.22}
$$

While the integrals are doable in the general case, the procedure is tedious and the final formulas are cumbersome. Therefore, for the moment we proceed as far as possible without using the explicit results of their evaluation.

3.3 The absorptive part $M^{(1)}_{\rm abs}(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c)$

We carry out the same procedure with the amplitude for the conjugate process. From Eq. (2.20),

$$
M^{(0)}\left(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}^{(n)}_a\bar{\ell}^{(n)}_c\right) = \Big[\bar{v}_a(k)\gamma^\mu(X'^* + Y'^*\gamma_5)v_a(q_1)\Big] \Big[\bar{v}_b(p)\gamma_\mu L v_\gamma(q_2)\Big]
\tag{3.23}
$$

which, using relations such as those give in Eqs. (3.15) and (3.14), can be written in the form

$$
M^{(0)}\left(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}^{(n)}_a\bar{\ell}^{(n)}_c\right) = \Big[\bar{u}_a(q_1)\gamma^\mu(X'^* - Y'^*\gamma_5)u_a(k)\Big] \Big[\bar{u}_c(q_2)\gamma_\mu R u_b(p)\Big] .
\tag{3.24}
$$

From Eq. (3.12), and using Eq. (3.16), we then obtain

$$
\begin{aligned}
M^{(1)}_{\rm abs}(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c) = \frac{ie^2}{2} \int dL \Big[\bar{u}_a(k')\gamma^\lambda(\not{q}_1 + m_a)\gamma^\mu(X'^* - Y'^*\gamma_5)u_a(k)\Big] \\
\times \Big[\bar{u}_c(p')\gamma_\lambda(\not{q}_2 + m_c)\gamma_\mu R u_b(p)\Big] .
\end{aligned}
\tag{3.25}
$$

By comparison, it is immediately seen that the amplitude for this process is obtained from the formulas for the direct process by making the substitutions

$$
\begin{aligned}
X' &\to X'^* \\
Y' &\to Y'^* \\
\gamma_5 &\to -\gamma_5 .
\end{aligned}
\tag{3.26}
$$

4 The difference in the rates

If we write the total amplitude in the form

$$
\begin{aligned}
M &= M^{(0)}(\ell_a\ell_b \to \ell_a\ell_c) + M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c) \\
\overline{M} &= M^{(0)}(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c) + M^{(1)}_{\rm abs}(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c) ,
\end{aligned}
\tag{4.1}
$$

the quantity in which we are interested is the difference

$$
\begin{aligned}
\left\langle |M|^2 \right\rangle - \left\langle |\overline{M}|^2 \right\rangle = 2\,{\rm Re}\, \left\langle M^{(0)*}(\ell_a\ell_b \to \ell_a\ell_c) M^{(1)}_{\rm abs}(\ell_a\ell_b \to \ell_a\ell_c) \right\rangle \\
- 2\,{\rm Re}\, \left\langle M^{(0)*}(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c) M^{(1)}_{\rm abs}(\bar{\ell}_a\bar{\ell}_b \to \bar{\ell}_a\bar{\ell}_c) \right\rangle ,
\end{aligned}
\tag{4.2}
$$

where the angle bracket notation indicate the operation to sum and average over the final and initial spins, respectively. In this expression we have made use of the fact that, after that operation is made, the terms without the absorptive part cancel out. Using Eq. (3.21), and denoting $M^{(0)}(\ell_a\ell_b \to \ell_a\ell_c)$ by simply $M^{(0)}$, we can write

$$\left\langle |M|^2 \right\rangle - \left\langle |\overline{M}|^2 \right\rangle = R - \bar{R}\,, \tag{4.3}$$

where

$$R \equiv -e^2 \sum_{i=1}^{4} \mathrm{Im}\left\langle M^{(0)*} M_i^{(1)} \right\rangle \tag{4.4}$$

and $\bar{R}$ is obtained from R by making the substitutions indicated in Eq. (3.26). We now compute the various terms in R.

4.1 1st term

Averaging over initial spins and summing over final spins, we obtain

$$\begin{aligned}
\left\langle M^{(0)*} M_1^{(1)} \right\rangle &= \frac{1}{4} m_a m_c I^{(0)}\,\mathrm{Tr}\left[(\not{p} + m_b)\gamma_\alpha L(\not{p}' + m_c)\gamma_\lambda\gamma_\mu L\right] \\
&\times\ \mathrm{Tr}\left[(\not{k} + m_a)\gamma^\alpha (X'^* + Y'^*\gamma_5)(\not{k}' + m_a)\gamma^\lambda\gamma^\mu (X' + Y'\gamma_5)\right] \qquad (4.5)\\
&= \frac{1}{4} m_a^2 m_c^2 I^{(0)}\,\mathrm{Tr}\left(\not{p}\gamma_\mu\gamma_\lambda\gamma_\rho L\right) \\
&\times\ \mathrm{Tr}\left[\not{k}\gamma^\mu\gamma^\lambda\gamma^\rho\left(|X'|^2 + |Y'|^2 + X'^*Y'\gamma_5 + Y'^*X'\gamma_5\right)\right. \\
&\quad \left. + \gamma^\mu\not{k}'\gamma^\lambda\gamma^\rho\left(|X'|^2 + |Y'|^2 + X'^*Y'\gamma_5 - Y'^*X'\gamma_5\right)\right] \qquad (4.6)
\end{aligned}$$

The traces are easily evaluated with the help of the formulas

$$\begin{aligned}
\mathrm{Tr}\,\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta &= 4C_{\alpha\beta\gamma\delta} \\
\mathrm{Tr}\,\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_5 &= -4i\epsilon_{\alpha\beta\gamma\delta}\,,
\end{aligned} \tag{4.7}$$

where

$$C_{\alpha\beta\gamma\delta} = g_{\alpha\beta}g_{\gamma\delta} - g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma} \tag{4.8}$$

When Eq. (4.7) is used in (4.6), four terms are produced, which we schematically denote as CC, $C\epsilon$, ϵC and $\epsilon\epsilon$, indicating which factor C or ϵ they contain from each of the two traces that appear. It is easy to see that the term CC is real, while the terms $C\epsilon$ and ϵC are zero after contracting the corresponding Lorentz indices. Only the term $\epsilon\epsilon$ has a non-zero imaginary part, and a little bit of algebra shows that

$$\mathrm{Im}\left\langle M^{(0)*} M_1^{(1)} \right\rangle = 24 m_a^2 m_c^2 I^{(0)} k' \cdot p\,\mathrm{Im}\,(X'^*Y')\,. \tag{4.9}$$

4.2 2nd term

For the remaining terms, it is useful to use the identity

$$\gamma^\lambda\gamma^\rho\gamma^\mu = C^{\lambda\rho\mu\alpha}\gamma_\alpha + i\epsilon^{\lambda\rho\mu\alpha}\gamma_\alpha\gamma_5\,, \tag{4.10}$$

where $C^{\lambda\rho\mu\alpha}$ is defined in Eq. (4.8). It then follows that

$$\gamma^\lambda\gamma^\rho\gamma^\mu L = \Big(C^{\lambda\rho\mu\alpha} - i\epsilon^{\lambda\rho\mu\alpha}\Big)\gamma_\alpha L\,, \tag{4.11}$$

and

$$\begin{aligned}\Big\langle M^{(0)*}M_2^{(1)}\Big\rangle = {} & \frac{1}{4}m_a I^{(2)\rho}\Big(C_{\lambda\rho\mu\alpha} - i\epsilon_{\lambda\rho\mu\alpha}\Big) \\ & \times \mathrm{Tr}\Big[(\not k + m_a)\gamma^\beta(X'^* + Y'^*\gamma_5)(\not k' + m_a)\gamma^\lambda\gamma^\mu(X' + Y'\gamma_5)\Big] \\ & \times \mathrm{Tr}\Big[(\not p + m_b)\gamma_\beta L(\not p' + m_c)\gamma^\alpha L\Big]\,. \end{aligned} \tag{4.12}$$

It is not difficult to see that all contributions involving the C term from the first parenthesis are real. Among the other terms that are not necessarily real, some are zero identically after contracting the Lorentz indices and there are others that, while not zero identically, are proportional to either one of the following factors

$$\epsilon^{\lambda\rho\sigma\tau} I^{(2)}_\rho p_\sigma p'_\tau k_\lambda\,, \qquad \epsilon^{\lambda\rho\sigma\tau} I^{(2)}_\rho p_\sigma p'_\tau k'_\lambda\,. \tag{4.13}$$

Since the integral $I^{(2)}_\rho$ is a vector that depends on p' and k', it is proportional to either p'_ρ or k'_ρ. Whence all such terms eventually appear contracted in the form

$$\epsilon^{\lambda\rho\sigma\tau} k_\rho p_\sigma p'_\tau k'_\lambda\,, \tag{4.14}$$

and in the end yield zero by momentum conservation. In summary, in the first trace that appears in Eq. (4.12), only the terms that contain the combination $X'^*Y' - X'Y'^*$ contribute to the difference in the rates, and a little bit of algebra yields

$$\mathrm{Im}\,\Big\langle M^{(0)*}M_2^{(1)}\Big\rangle = -8m_a^2\,\mathrm{Im}\,(X'^*Y')I^{(2)\rho}\Big[k'_\rho p\cdot p' + p_\rho k'\cdot p' + p'_\rho k'\cdot p\Big]\,. \tag{4.15}$$

4.3 3rd term

Using Eq. (4.10) once more, by straightforward algebraic manipulations we obtain

$$\begin{aligned}\Big\langle M^{(0)*}M_3^{(1)}\Big\rangle = {} & \frac{1}{4}m_c I^{(1)}_\rho \times \mathrm{Tr}\Big[(\not p + m_b)\gamma_\beta L(\not p' + m_c)\gamma_\lambda\gamma_\mu L\Big] \\ & \times\Big(C^{\lambda\rho\mu\alpha}\,\mathrm{Tr}\Big[(\not k + m_a)\gamma^\beta(X'^* + Y'^*\gamma_5)(\not k' + m_a)\gamma_\alpha(X' + Y'\gamma_5)\Big] \\ & + i\epsilon^{\lambda\rho\mu\alpha}\,\mathrm{Tr}\Big[(\not k + m_a)\gamma^\beta(X'^* + Y'^*\gamma_5)(\not k' + m_a)\gamma_\alpha(Y' + X'\gamma_5)\Big]\Big)\,. \end{aligned} \tag{4.16}$$

The remaining manipulations and arguments are similar to those that lead to Eq. (4.15), and in this case they yield

$$\mathrm{Im}\,\Big\langle M^{(0)*}M_3^{(1)}\Big\rangle = 24m_a^2 m_c^2\,\mathrm{Im}\,(X'^*Y')I^{(1)}_\rho p^\rho\,. \tag{4.17}$$

4.4 4th term

It is useful to notice that the expression for $M_4^{(1)}$ can be simplified by using the identity in Eq. (4.10), and using then the formulas

$$\begin{aligned} C^{\lambda\rho\mu\alpha}C_{\lambda\nu\mu\beta} &= 2(\delta^\rho_\nu\delta^\alpha_\beta + \delta^\alpha_\nu\delta^\rho_\beta) \\ \epsilon^{\lambda\rho\mu\alpha}\epsilon_{\lambda\nu\mu\beta} &= -2(\delta^\rho_\nu\delta^\alpha_\beta - \delta^\alpha_\nu\delta^\rho_\beta)\,. \end{aligned} \tag{4.18}$$

Thus we obtain

$$
\begin{aligned}
M_4^{(1)} &= 4\Big[\bar{u}_c(p')\gamma^\alpha L u_b(p)\Big]\Big\{I_\beta^{(12)\beta}(X'-Y')\Big[\bar{u}_a(k')\gamma_\alpha L u_a(k)\Big] \\
&\quad + I_{\alpha\beta}^{(12)}(X'+Y')\Big[\bar{u}_a(k')\gamma^\beta R u_a(k)\Big]\Big\}\,,
\end{aligned} \tag{4.19}
$$

and then

$$
\begin{aligned}
\Big\langle M^{(0)*}M_4^{(1)}\Big\rangle &= \mathrm{Tr}\left[(\not{p}+m_b)\gamma_\mu L(\not{p}'+m_c)\gamma^\alpha L\right] \\
&\quad\times\Big\{I_\beta^{(12)\beta}(X'-Y')\,\mathrm{Tr}\left[(\not{k}+m_a)\gamma^\mu(X'^*+Y'^*\gamma_5)(\not{k}'+m_a)\gamma_\alpha L\right] \\
&\quad + I_{\alpha\beta}^{(12)}(X'+Y')\,\mathrm{Tr}\Big[(\not{k}+m_a)\gamma^\mu(X'^*+Y'^*\gamma_5)(\not{k}'+m_a)\gamma^\beta R\Big]\Big\}\,.
\end{aligned} \tag{4.20}
$$

When we carry out the traces and perform the contractions, the terms that do not have the factor of m_a^2 turn out to be real, proportional either to $|X'-Y'|^2$ or to $|X'+Y'|^2$. The only terms that contribute to the rate difference are those that have the factor of m_a^2, and by the same manipulations that lead to Eq. (4.12) we find

$$
\mathrm{Im}\,\Big\langle M^{(0)*}M_4^{(1)}\Big\rangle = 8m_a^2\,\mathrm{Im}\,(X'^*Y')I_{\alpha\beta}^{(12)}\Big[p^\alpha p'^\beta + p'^\alpha p^\beta + g^{\alpha\beta}p\cdot p'\Big]\,. \tag{4.21}
$$

4.5 The sum

Summarizing the formulas that we have obtained, we can now write

$$
R = -e^2\,\mathrm{Im}\,(X'^*Y')\left[Z_1+Z_2+Z_3+Z_4\right]\,, \tag{4.22}
$$

where

$$
\begin{aligned}
Z_1 &= 24m_a^2 m_c^2 I^{(0)}p\cdot k'\,, \\
Z_2 &= -8m_a^2 I^{(2)\rho}\Big[p_\rho(p'\cdot k') + k'_\rho(p\cdot p') + p'_\rho(p\cdot k')\Big]\,, \\
Z_3 &= 24m_a^2 m_c^2 I_\rho^{(1)}p^\rho\,, \\
Z_4 &= 8m_a^2 I_{\alpha\beta}^{(12)}\Big[p^\alpha p'^\beta + p'^\alpha p^\beta + g^{\alpha\beta}p\cdot p'\Big]\,.
\end{aligned} \tag{4.23}
$$

It is now obvious that $\bar{R}$, which is to be computed similarly but with the substitution indicated in Eq. (3.26), is given by $\bar{R}=-R$, and therefore

$$
\Big\langle|M|^2\Big\rangle - \Big\langle|\overline{M}|^2\Big\rangle = 2R\,. \tag{4.24}
$$

Clearly the Z_i, and consequently the CP violating effects given by R, vanish in the limit that all the charged lepton masses are taken to be zero. We then consider the quantities Z_i evaluated to the lowest order in the charged lepton masses; i.e., we keep only those terms that contain two powers of the charged lepton mass. At this order, Z_1 and Z_3 do not contribute. Since Z_2 and Z_4 already have an explicit factor m_a^2, we evaluate the other kinematic factors, for massless particles. As shown in Appendix D, in this limit the relevant integrals are given by

$$
\begin{aligned}
I_\mu^{(2)} &= B_0P_\mu - B_1Q_\mu \\
I_{\mu\nu}^{(12)} &= \frac{1}{4}\Big[(B_0-B_2)sg_{\mu\nu} + (2B_0-B_0+B_2)P_\mu P_\nu \\
&\quad + 2B_1(Q_\mu P_\nu - P_\mu Q_\nu) - (3B_2-B_0)Q_\mu Q_\nu\Big]\,,
\end{aligned} \tag{4.25}
$$

where

$$
\begin{aligned}
P &= k' + p' \,, \\
Q &= k' - p' \,, \\
s &= P^2 \,,
\end{aligned}
\tag{4.26}
$$

and

$$
B_n = -\frac{1}{16\pi s} \int_{-1}^{+1} d\xi \, \frac{\xi^n}{1-\xi} \,. \tag{4.27}
$$

Thus

$$
\begin{aligned}
Z_2 &= -8 m_a^2 s \left[B_0 p \cdot P - B_1 p \cdot Q \right] \\
Z_4 &= 4 m_a^2 s \left[2 B_0 p \cdot P - (B_0 + B_2) p \cdot Q \right] \,,
\end{aligned}
\tag{4.28}
$$

and we finally obtain

$$
Z \equiv Z_2 + Z_4 = 4 m_a^2 s [2B_1 - B_0 - B_2] \, p \cdot Q + O(m_\ell^4 s) \,. \tag{4.29}
$$

It is reassuring to observe that, while the integrals B_n defined in Eq. (4.27) are (infrared) divergent individually, the combination that appears in Eq. (4.29) is divergent-free, and its value is given by

$$
2B_1 - B_0 - B_2 = \frac{1}{8\pi s} \,. \tag{4.30}
$$

Thus, from Eqs. (4.22) and (4.29),

$$
R = -\frac{e^2}{2\pi} m_a^2 \, \mathrm{Im}\, (X'^* Y')(p \cdot Q) \,, \tag{4.31}
$$

and finally, from Eq. (4.24),

$$
\left\langle |M|^2 \right\rangle - \left\langle |\overline{M}|^2 \right\rangle = -\frac{e^2}{\pi} m_a^2 \, \mathrm{Im}\, (X'^* Y')(p \cdot Q) \,. \tag{4.32}
$$

On the other hand, the leading term of the amplitude squared is straightforward to calculate and yields

$$
\left\langle |M^{(0)}|^2 \right\rangle = \left\langle |\overline{M}^{(0)}|^2 \right\rangle = 4|X' + Y'|^2 (k \cdot p')(k' \cdot p) + 4|X' - Y'|^2 (k \cdot p)(k' \cdot p') \,, \tag{4.33}
$$

which determines the total rate. Taking the massless limit approximation, and using Eq. (4.32), we then have

$$
\frac{1}{\Gamma} \left[\frac{d\Gamma}{d(\cos\theta)} - \frac{d\bar{\Gamma}}{d(\cos\theta)} \right] = -\frac{e^2}{4\pi} \frac{(m_a^2/s) \, \mathrm{Im}\, (X'^* Y')}{|X' + Y'|^2 + (1/3)|X' - Y'|^2} \cos\theta \,, \tag{4.34}
$$

where θ is the angle between $\hat{k}$ and $\hat{k}'$, in both the direct and the conjugate processes.

5 Discussion and Conclusions

Using Eq. (2.21) and the formulas given in Eqs. (2.10) and (2.16), the CP violating quantity $\mathrm{Im}\,(X'^*Y')$ that appears in Eq. (4.34) is proportional to

$$\delta \equiv (X+Y) \sum_{i,j,k} \Big\{ f_A^{(ij)} f_Z^{(k)} \Big[\mathrm{Im}\,(t_{bjck}) |V_{ai}|^2 + \mathrm{Im}\,(t_{aicj} t_{bjck}) |V_{cj}|^{-2} \Big] + f_B^{(ij)} f_Z^{(k)}\, \mathrm{Im}\,(s_{aij} s_{bik} s_{ckj}) \Big\}, \tag{5.1}$$

where X, Y are the neutral-current couplings defined in Eq. (2.18), $f_{A,B}^{(ij)}$ and $f_Z^{(k)}$ are the kinematic factors defined in Eqs. (2.11) and (2.17) while the coefficients t_{bjck} and s_{aij} are given by

$$\begin{aligned} t_{aibj} &= V_{ai} V_{bj} V_{aj}^* V_{bi}^*\,, \\ s_{aij} &= V_{ai} V_{aj}^* K_i^* K_j\,. \end{aligned} \tag{5.2}$$

The t coefficients are the rephasing-invariant parameters of the lepton sector that are analogous to those introduced for the quark sector in Refs. [15, 16]. These parameters occur in a purely lepton-number conserving theory. In fact, if lepton-number is conserved in the theory, the diagrams in Fig. 1B do not exist, so that we can put $f_B^{(ij)} = 0$. In this case, Eq. (5.1) clearly shows that only the t-invariants appear in the CP-violating part of the amplitude.

On the other hand, the s coefficients are precisely the rephasing invariants introduced in Ref. [1] to accommodate the Majorana neutrinos. As shown in there, and further studied in Ref. [2], the s coefficients form a suitable set of rephasing-invariant parameters for describing the CP violating effects due to the Majorana nature of the neutrinos. In fact, the dependence of δ on the product of three s parameters, as indicated in Eq. (5.1), was anticipated in Ref. [2] [e.g., Eqs. (3.27) and (3.37) of that paper].

Thus, the present calculation confirms the expectation that the extra CP violating phases that exist for Majorana neutrinos can appear in CP violating observables in processes that conserve total lepton number. Although the type of process that we have specifically considered (e.g., $e + \mu \to e + \tau$) is not a realistic one at present, related processes such as $\tau \to \bar{e} + e + \mu$ will show the same effect. The corresponding calculations for the latter kind of process is more involved than those presented here due to the three-body final state involved. Nevertheless, the present calculations, besides serving as a proof of concept, set the stage for considering such three-body decay process, and should prove to be technically useful in that context as well.

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A Calculation of the box diagrams

We calculate here the diagrams shown in Fig. 1, for arbitrary incoming and outgoing lepton flavors. We denote the momentum vectors by k_ℓ, with $\ell = a, b, c, d$ according to the labels assigned in the diagrams.

Diagram (1A)

Not counting the exchange diagrams, we have four diagrams altogether. However, those that involve either one or two unphysical Higgs in the internal lines are of order $1/M_W^6$, and we neglect them. Therefore, in terms of the couplings of the charged current $j_\mu^{(W)} = \sum_{a,i} V_{ai}\bar{\ell}_a\gamma_\mu L\nu_i$,

$$\begin{aligned} iM_A^{(abcd)} &= \left(\frac{-ig}{\sqrt{2}}\right)^4 \sum_{i,j} \int \frac{d^4q}{(2\pi)^4} \left[V_{dj}V_{bj}^*\bar{u}_d(k_d)\gamma_\rho LiS_{\nu_j}(k_d-k_a+q)\gamma_\nu Lu_b(k_b)\right] \\ &\times \left[V_{ci}V_{ai}^*\bar{u}_c(k_c)\gamma_\mu LiS_{\nu_i}(q)\gamma_\lambda Lu_a(k_a)\right] \left(\frac{-ig^{\mu\nu}}{(k_c-q)^2-M_W^2}\right)\left(\frac{-ig^{\lambda\rho}}{(k_a-q)^2-M_W^2}\right) . \end{aligned} \tag{A.1}$$

This expression can be written in the form

$$iM_A^{(abcd)} = \left(\frac{g}{\sqrt{2}}\right)^4 \sum_{i,j}(V_{dj}V_{bj}^*V_{ci}V_{ai}^*)I_\lambda^\rho \left[\bar{u}_d(k_d)\gamma_\nu\gamma_\rho\gamma_\mu Lu_b(k_b)\right]\left[\bar{u}_c(k_c)\gamma^\mu\gamma^\lambda\gamma^\nu Lu_a(k_a)\right] , \tag{A.2}$$

where

$$I_\lambda^\rho \equiv \int \frac{d^4q}{(2\pi)^4} \frac{q^\rho q_\lambda}{[(k_c-q)^2-M_W^2][(k_a-q)^2-M_W^2][q^2-m_{\nu_i}^2][(k_d-k_a+q)^2-m_{\nu_j}^2]} . \tag{A.3}$$

Neglecting terms $O(1/M_W^6)$,

$$I_{\rho\lambda} = \frac{1}{4}g_{\rho\lambda}\left(\frac{i}{16\pi^2M_W^2}\right)f_A^{(ij)} \tag{A.4}$$

where $f_A^{(ij)}$ is given in Eq. (2.11) in the text. Finally, using the identity

$$\left[\bar{u}_d(k_d)\gamma_\nu\gamma_\lambda\gamma_\mu Lu_b(k_b)\right]\left[\bar{u}_c(k_c)\gamma^\mu\gamma^\lambda\gamma^\nu Lu_a(k_a)\right] = 16\left[\bar{u}_d(k_d)\gamma_\lambda Lu_b(k_b)\right]\left[\bar{u}_c(k_c)\gamma^\lambda Lu_a(k_a)\right] , \tag{A.5}$$

we arrive at

$$M_A^{(abcd)} = \frac{g^4}{64\pi^2M_W^2}\sum_{i,j}(V_{ai}^*V_{bj}^*V_{ci}V_{dj})(4f_A^{(ij)})\mathcal{M}_W^{(abcd)} \tag{A.6}$$

where

$$\mathcal{M}_W^{(abcd)} \equiv [\bar{u}_d(k_d)\gamma^\mu Lu_b(k_b)][\bar{u}_c(k_c)\gamma_\mu Lu_a(k_a)] . \tag{A.7}$$

The contribution to the physical amplitude, including the exchange term, is given by

$$M_A(\ell_a\ell_b\to\ell_c\ell_d) = M_A^{(abcd)} - M_A^{(abdc)} . \tag{A.8}$$

Using the fact that $\mathcal{M}_W^{(abcd)} = -\mathcal{M}_W^{(abdc)}$, which follows from a Fierz transformation, the contribution to the physical amplitude is then

$$M_A(\ell_a\ell_b\to\ell_c\ell_d) = \lambda_A^{(abcd)}\mathcal{M}_W^{(abcd)} , \tag{A.9}$$

with $\lambda_A^{(abcd)}$ as defined in Eq. (2.10) in the text.

Diagram (1B)

As with the diagrams 1A, there are four diagrams not counting the exchange diagrams, and those that involve either one or two unphysical Higgs in the internal lines are of order $1/M_W^6$. Therefore, to order $1/M_W^4$,

$$
\begin{aligned}
iM_B^{(abcd)} &= \left(\frac{g}{\sqrt{2}}\right)^4 \sum_{i,j} \int \frac{d^4q}{(2\pi)^4} \Big[\bar{u}_c(k_c)(-i\gamma_\nu L V_{cj}) i K_j^{*\,2} S_{\nu_j}(k_a - k_d - q)(i\gamma_\rho R V_{dj}) v_d(k_d)\Big] \\
&\times \Big[\bar{v}_b(k_b)(i\gamma_\mu R V_{bi}^*) i K_i^2 S_{\nu_i}(q)(-i\gamma_\lambda L V_{ai}^*) u_a(k_a)\Big] \left[\frac{-ig^{\mu\nu}}{(k_b+q)^2 - M_W^2}\right] \left[\frac{-ig^{\lambda\rho}}{(k_a-q)^2 - M_W^2}\right],
\end{aligned}
\tag{A.10}
$$

where we have used the Majorana condition of Eq. (2.13). Only the mass terms of the neutrino propagators contribute due to the L and R factors on opposite sides of the propagator, and we obtain

$$
\begin{aligned}
iM_B^{(abcd)} &= \left(\frac{g}{\sqrt{2}}\right)^4 \sum_{i,j} \left(m_{\nu_i} m_{\nu_j} K_i^2 K_j^{*\,2} V_{cj} V_{dj} V_{bi}^* V_{ai}^*\right) iI_B \\
&\quad \times \Big[\bar{u}_c(k_c)\gamma^\lambda \gamma^\rho R v_d(k_d)\Big] \Big[\bar{v}_b(k_b)\gamma_\lambda \gamma_\rho L u_a(k_a)\Big],
\end{aligned}
\tag{A.11}
$$

where

$$
iI_B \equiv \int \frac{d^4q}{(2\pi)^4} \frac{1}{[(k_b+q)^2 - M_W^2][(k_a-q)^2 - M_W^2][q^2 - m_{\nu_i}^2][(k_a-k_d-q)^2 - m_{\nu_j}^2]} . \tag{A.12}
$$

Neglecting terms $O(1/M_W^6)$, the explicit evaluation of I_B yields

$$
I_B = \frac{1}{16\pi^2 M_W^2} \left\{ \frac{r_i \log r_i}{r_j - r_i} - r_i + (i \leftrightarrow j) \right\} . \tag{A.13}
$$

Finally, by a Fierz transformation,

$$
\Big[\bar{u}_c(k_c)\gamma^\lambda \gamma^\rho R v_d(k_d)\Big] \Big[\bar{v}_b(k_b)\gamma_\lambda \gamma_\rho L u_a(k_a)\Big] = 2\Big[\bar{u}_d(k_d)\gamma^\lambda L u_b(k_b)\Big] \Big[\bar{u}_c(k_c)\gamma_\lambda L u_a(k_a)\Big], \tag{A.14}
$$

the details of which are shown in Appendix B. Thus Eq. (A.11) reduces to

$$
M_B^{(abcd)} = \frac{g^4}{64\pi^2 M_W^2} \sum_{i,j} \left(K_i^2 K_j^{*\,2} V_{ai}^* V_{bi}^* V_{cj} V_{dj}\right) (2 f_B^{(ij)}) \mathcal{M}_W^{(abcd)} , \tag{A.15}
$$

where $\mathcal{M}_W^{(abcd)}$ has been defined in Eq. (A.7). As with the previous diagram, the contribution to the physical amplitude, including the exchange term, is given by

$$
\begin{aligned}
M_B(\ell_a \ell_b \to \ell_c \ell_d) &= M_B^{(abcd)} - M_B^{(abdc)} \\
&= \lambda_B^{(abcd)} \mathcal{M}_W^{(abcd)} ,
\end{aligned}
\tag{A.16}
$$

with $\lambda_B^{(abcd)}$ as defined in Eq. (2.10).

B Fierz transformations

Fierz transformations concern products of two fermion bilinears. For arbitrary spinors w_1, w_2, w_3 and w_4, we denote these products collectively as e_i with $i = S, V, T, A, P$. In other words,

$$e_i = [\bar{w}_1 \Gamma^i w_2][\bar{w}_3 \Gamma_i w_4] , \tag{B.1}$$

with

$$\Gamma_i = (1, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu \gamma_5, \gamma_5) , \tag{B.2}$$

and Γ^i being the same things with contravariant Lorentz indices. Then we will denote by e'_i the same quantities, but with $w_2 \leftrightarrow w_4$; i.e.,

$$e'_i = [\bar{w}_1 \Gamma^i w_4][\bar{w}_3 \Gamma_i w_2] . \tag{B.3}$$

The basic Fierz identity is the relation between the two sets of bilinears, $\{e_i\}$ and $\{e'_i\}$,

$$e_i = \sum_j F_{ij} e'_j , \tag{B.4}$$

where

$$F = \frac{1}{4} \begin{pmatrix} 1 & 1 & \frac{1}{2} & -1 & 1 \\ 4 & -2 & 0 & -2 & -4 \\ 12 & 0 & -2 & 0 & 12 \\ -4 & -2 & 0 & -2 & 4 \\ 1 & -1 & \frac{1}{2} & 1 & 1 \end{pmatrix} . \tag{B.5}$$

To prove Eq. (A.14) from here, first note that the identity

$$\gamma^\lambda \gamma^\rho = g^{\lambda\rho} - i\sigma^{\lambda\rho} \tag{B.6}$$

can be used to write the left hand side of Eq. (A.14) as $4e_S - e_T$ in the notation of Eq. (B.1), with

$$\begin{aligned} w_1 &= u_{Lc}(k_c) \\ w_2 &= v_{Rd}(k_d) \\ w_3 &= v_{Rb}(k_b) \\ w_4 &= u_{La}(k_a) . \end{aligned} \tag{B.7}$$

The relevant Fierz formula for us now is

$$\begin{aligned} 4e_S - e_T &= \left(e'_S + e'_P + \frac{1}{2} e'_T + e'_V - e'_A \right) - \left(3e'_S + 3e'_P - \frac{1}{2} e'_T \right) \\ &= -2(e'_S + e'_P) + e'_T + e'_V - e'_A . \end{aligned} \tag{B.8}$$

Substituting the spinors a-d given above, it turns out that $e'_{S,P,T} = 0$, while

$$\begin{aligned} -e'_A = e'_V &= \left[\bar{u}_{Lc}(k_c) \gamma^\lambda u_{La}(k_a) \right] \left[\bar{v}_{Rb}(k_b) \gamma_\lambda v_{Rd}(k_d) \right] \\ &= \left[\bar{u}_{Lc}(k_c) \gamma^\lambda u_{La}(k_a) \right] \left[\bar{u}_{Ld}(k_d) \gamma_\lambda u_{Lb}(k_b) \right] , \end{aligned} \tag{B.9}$$

using Eq. (3.15) and similar relations in the last step. This gives Eq. (A.14).

C Derivation of Eq. (3.5)

In order to make the discussion generally applicable and not tied to any particular channel, we consider the process labeled as

$$\ell_a(k_a) + \ell_b(k_b) \rightarrow \ell_c(k_c) + \ell_d(k_d)\,. \tag{C.1}$$

Thus, this includes processes such as $\ell_a \ell_b \rightarrow \ell_a \ell_a$ and $\ell_a \ell_b \rightarrow \ell_a \ell_c$. The treatment for the crossed processes such as $\bar{\ell}_a \bar{\ell}_b \rightarrow \bar{\ell}_a \bar{\ell}_a$ is similar, with the appropriate modifications dictated by the usual substitution (crossing) rules. The amplitude $M^{(0)}$, determined from the diagrams that are schematically represented in Fig. 3, can be expressed in the form

$$iM^{(0)} = i \sum_{A,B} g_{AB} I_{AB}(k_a, k_b, k_c, k_d) \Big[\bar{u}_c(k_c)\Gamma_A u_a(k_a)\Big] \Big[\bar{u}_d(k_d)\Gamma_B u_b(k_b)\Big]\,, \tag{C.2}$$

where each g_{AB} denotes a product of coupling constants, each I_{AB} is Feynman integral which is real, and the Γ_A are the generalized Dirac-Pauli matrices. In the most general case, the amplitude can be brought to this form by making the appropriate Fierz transformations. With this in place, the amplitude for $M^{(1)}$, determined from the diagrams represented in Fig. 4, is given by

$$\begin{aligned} iM^{(1)} = & \int \frac{d^4q}{(2\pi)^4} i \sum_{A,B} g_{AB} I_{AB}(k_a, k_b, k_c + q, k_d - q) i D_F^{\nu\mu}(q) \\ & \times [\bar{u}_c(k_c)(-ie_\gamma \gamma_\mu) i S_{Fc}(k_c + q)\Gamma_A u_a(k_a)] \\ & \times [\bar{u}_d(k_d)(-ie_d \gamma_\nu) i S_{Fd}(k_d - q)\Gamma_B u_b(k_b)]\ . \end{aligned} \tag{C.3}$$

Our task is to determine the absorptive part of $M^{(1)}$.

Since the integrals I_{AB} are real, the absorptive part can arise only from the denominators of the lepton propagators in Eq. (C.3), which we write in the form

$$D = (q^0 - a)(q^0 - a')(q^0 - b)(q^0 - b')\,, \tag{C.4}$$

where

$$\begin{aligned} a &= E^{(c)}_{k_c+q} - E^{(c)}_{k_c} - i\epsilon \\ a' &= -(E^{(c)}_{k_c+q} + E^{(c)}_{k_c}) + i\epsilon \\ b &= E^{(d)}_{k_d} + E^{(d)}_{k_d-q} - i\epsilon \\ b' &= E^{(d)}_{k_d} - E^{(d)}_{k_d-q} + i\epsilon\,, \end{aligned} \tag{C.5}$$

with

$$E^{(\ell)}_p = \sqrt{\vec{p}^{\,2} + m_\ell^2}\,. \tag{C.6}$$

Therefore, we rewrite Eq. (C.3) as

$$iM^{(1)} = \int \frac{d^4q}{(2\pi)^4} \frac{1}{D} i\mathcal{M}^{(1)}\,, \tag{C.7}$$

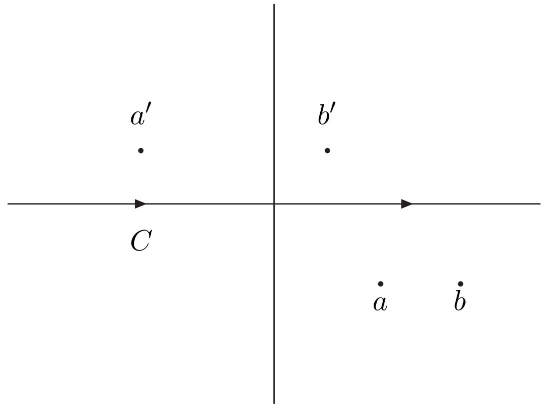

Figure 5: Original path of integration in the q^0 plane.

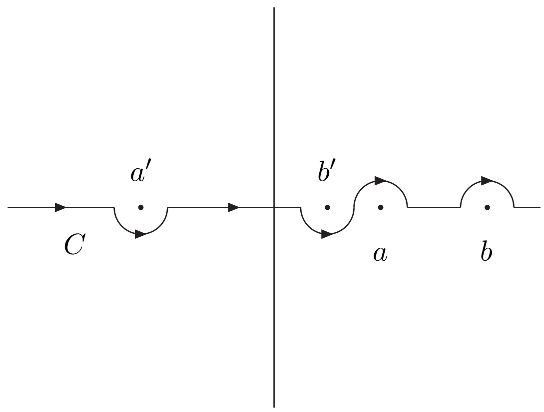

Figure 6: Deformed path of integration in the q^0 plane to avoid the poles when $\epsilon \to 0$.

where

$$\begin{aligned} i\mathcal{M}^{(1)} &= i \sum_{A,B} g_{AB} I_{AB}(k_a, k_b, k_c + q, k_d - q) i D_F^{\nu\mu}(q) \\ &\times [\bar{u}_c(k_c)(-ie_c\gamma_\mu) i(\not{k}_c + \not{q} + m_c)\Gamma_A u_a(k_a)] \\ &\times [\bar{u}_d(k_d)(-ie_d\gamma_\nu) i(\not{k}_d - \not{q} + m_d)\Gamma_B u_b(k_b)] \ . \end{aligned} \tag{C.8}$$

From a mathematical point of view, the absorptive part of $M^{(1)}$ arises from the fact that, as a function of q^0, the integrand in Eq. (C.7) has poles at the points indicated in Eq. (C.5), and illustrated schematically in Fig. 5. As $\epsilon \to 0$ those poles lie on the real axis but, as long as the path of integration can be deformed such that it avoids the poles, the resulting integral is real and the absorptive part of $M^{(1)}$ is zero. This is what happens for all the kinematic configurations in which none of the poles coincide with another one, as illustrated in Fig. 6. However, if the kinematic configuration is such that, in the limit $\epsilon \to 0$, one of the poles that lie above the real axis coincide with one that lies below the real axis, then the path of integration is *pinched* between the two points, and it cannot be deformed to avoid the poles. In this case the amplitude will develop an absorptive part. This side of the coin is also illustrated in Fig. 6, if we consider the case that $b' = a$. As a matter of fact, from the condition that $E_p^{(\ell)}$ is a positive quantity, it follows from Eq. (C.5) that this is the only possible *pinch condition*.

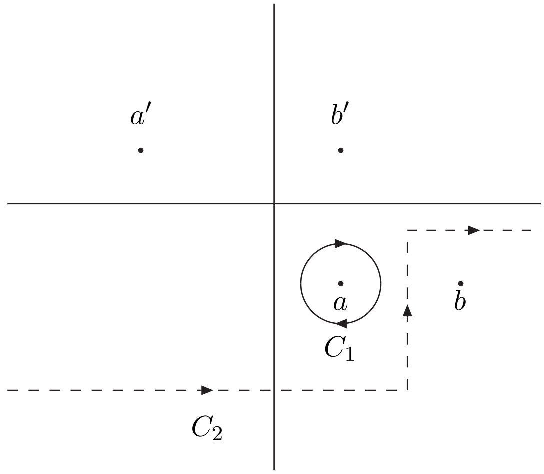

Figure 7: Deformed path of integration in the q^0 plane.

In order to isolate the contribution to $M^{(1)}$ in this situation we proceed as follows. By Cauchy's theorem, we can deform the original path of integration as shown in Fig. 7. The virtue of this is that we can write

$$iM^{(1)} = \left(\int \frac{d^4q}{(2\pi)^4}\frac{1}{D}i\mathcal{M}^{(1)}\right)_{C_1} + \left(\int \frac{d^4q}{(2\pi)^4}\frac{1}{D}i\mathcal{M}^{(1)}\right)_{C_2} . \tag{C.9}$$

By the same argument that we have explained above, it now follows that the integral over the path C_2 does not produce an absorptive part because the path cannot be pinched (i.e., there is no kinematic configuration for which b can become equal to a' or b'). Therefore, the integral over C_2 contributes only to the dispersive part of the amplitude and we neglect it. The integral over C_1 on the other hand, can be evaluated by the method of residues, and therefore

$$iM^{(1)} = -i\int \frac{d^3q}{(2\pi)^3}\frac{1}{D'}i\mathcal{M}^{(1)} , \tag{C.10}$$

where

$$D' = 2E^{(c)}_{k_c+q}\left[E^{(c)}_{k_c} + E^{(d)}_{k_d} - E^{(c)}_{k_c+q} + E^{(d)}_{k_d-q}\right]\left[E^{(c)}_{k_c} + E^{(d)}_{k_d} - E^{(c)}_{k_c+q} - E^{(d)}_{k_d-q} + i\epsilon\right] . \tag{C.11}$$

Once again, this has a dispersive and an absorptive contribution. Retaining only the latter, which is obtained by the substitution $1/(x+i\epsilon) \to -i\pi\delta(x)$, we finally obtain

$$iM^{(1)} = -\pi\int \frac{d^3q}{(2\pi)^3}\frac{1}{2E^{(c)}_{k_c+q}}\frac{1}{2E^{(d)}_{k_d-q}}\delta(E^{(c)}_{k_c} + E^{(d)}_{k_d} - E^{(c)}_{k_c+q} - E^{(d)}_{k_d-q})i\mathcal{M}^{(1)} . \tag{C.12}$$

To write this in its final form, we put in the expression for $\mathcal{M}^{(1)}$

$$\begin{aligned} \vec{q}_c &\equiv \vec{k}_c + \vec{q} , \\ \vec{q}_d &\equiv \vec{k}_d - \vec{q} , \end{aligned} \tag{C.13}$$

and insert the factor

$$\int \frac{d^3q_c}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{q}_c - \vec{k}_c - \vec{q})\int \frac{d^3q_d}{(2\pi)^3}(2\pi)^3\delta^{(3)}(\vec{q}_d - \vec{k}_d + \vec{q}) . \tag{C.14}$$

When these are substituted in Eq. (C.12), the integral over $d^3\vec{q}$ can be eliminated with the help of the delta functions, and we finally arrive at

$$iM^{(1)} = -\left(\frac{1}{2}\right)\int \frac{d^3q_c}{(2\pi)^3 2E^{(c)}_{q_c}} \int \frac{d^3q_d}{(2\pi)^3 2E^{(d)}_{q_d}} (2\pi)^4\delta(q_c+q_d-k_c-k_d) i\mathcal{M}^{(1)} \,, \qquad \text{(C.15)}$$

with $\mathcal{M}^{(1)}$ expressed in the form

$$\begin{aligned} i\mathcal{M}^{(1)} &= i\sum_{A,B}\left(g_{AB}I_{AB}(k_a,k_b,q_c,q_d)\right) iD_F^{\nu\mu}(q_c-k_c) \\ &\quad \times \left[\bar{u}_c(k_c)(-ie_c\gamma_\mu)i(\not{q}_c+m_c)\Gamma_A u_a(k_a)\right] \\ &\quad \times \left[\bar{u}_d(k_d)(-ie_d\gamma_\nu)i(\not{q}_d+m_d)\Gamma_B u_b(k_b)\right] . \end{aligned} \qquad \text{(C.16)}$$

Using the relation

$$(\not{q}_\ell + m_\ell) = \sum_s u_\ell(q_\ell)\bar{u}_\ell(q_\ell)\,, \qquad \text{(C.17)}$$

it is easily seen that the result given in Eq. (C.15) is equivalent to the formula quoted in Eq. (3.5).

D Integrals over intermediate states

Here we consider the evaluation of the integrals defined in Eq. (3.20), the results of which are quoted in Eq. (4.25). For the reasons mentioned in the text, we take all the lepton masses to be zero.

The measure dL, defined in Eq. (3.19), contains the factor

$$(k'-q_1)^2 = -\frac{1}{2}s(1-\xi)\,, \qquad \text{(D.1)}$$

where ξ is the cosine of the angle between $\vec{q}_1$ and $\vec{k}'$, and s has been defined in Eq. (4.26). Then, for any integrand F, we can write

$$\int dL\, F = -\frac{1}{8\pi s}\int_{-1}^{+1} d\xi\, \frac{F}{1-\xi}\,, \qquad \text{(D.2)}$$

performing as many integrations as the delta function allows us. Further, complementing Eq. (4.26) it is convenient to define

$$\begin{aligned} P' &= q_1 + q_2 \\ Q' &= q_1 - q_2\,. \end{aligned} \qquad \text{(D.3)}$$

Noting that momentum conservation ensures that $P'_\mu = P_\mu$, it follows that

$$I^{(1)}_\mu = \frac{1}{2}\int dL\,(P'_\mu + Q'_\mu) = \frac{1}{2}\left(P_\mu J + J_\mu\right)\,, \qquad \text{(D.4)}$$

and similarly

$$I^{(2)}_\mu = \frac{1}{2}\left(P_\mu J - J_\mu\right)\,, \qquad \text{(D.5)}$$

where we define a new set of integrals

$$J = \int dL$$
$$J_{\mu_1\mu_2\cdots\mu_n} \equiv \int dL\, Q'_{\mu_1} Q'_{\mu_2} \cdots Q'_{\mu_n}\,. \tag{D.6}$$

For the integral with two indices, we can similarly write

$$I^{(12)}_{\mu\nu} = \frac{1}{4}\left(P_\mu P_\nu J + J_\mu P_\nu - P_\mu J_\nu - J_{\mu\nu}\right)\,. \tag{D.7}$$

Therefore, we only have to evaluate the integrals in Eq. (D.6).

The scalar integral J can be determined immediately,

$$J = -\frac{1}{8\pi s}\int_{-1}^{+1} d\xi \frac{1}{1-\xi} = 2B_0\,, \tag{D.8}$$

where B_n is defined in Eq. (4.27). For the others, notice that

$$P^{\mu_i} J_{\mu_1\mu_2\cdots\mu_n} = 0 \tag{D.9}$$

for any $i = 1, 2, \cdots n$. For the one-index integral, there is only one such relation

$$P^\mu J_\mu = 0\,, \tag{D.10}$$

which implies that

$$J_\mu = bQ_\mu\,, \tag{D.11}$$

for some invariant b. The invariant can be determined by contracting both sides with Q^μ, which gives

$$b = \frac{Q^\mu J_\mu}{Q^2}\,. \tag{D.12}$$

By explicit computation,

$$Q^\mu J_\mu = -2sB_1\,, \tag{D.13}$$

and using $Q^2 = -s$, we arrive at

$$J_\mu = 2B_1 Q_\mu\,. \tag{D.14}$$

Substituting this in Eq. (D.5) then yields the formula for $I^{(2)}_\mu$ quoted in Eq. (4.25). An analogous formula for $I^{(1)}_\mu$ follows from Eq. (D.4).

For the two-index integral $J_{\mu\nu}$ the relation in Eq. (D.9) now dictates the general form

$$J_{\mu\nu} = \lambda(P_\mu P_\nu - s g_{\mu\nu}) + \rho Q_\mu Q_\nu\,, \tag{D.15}$$

with the coefficients being easily determined by contracting with $g^{\mu\nu}$ and by $Q^\mu Q^\nu$. The equations obtained this way are

$$s^2\lambda + s^2\rho = Q^\mu Q^\nu J_{\mu\nu}$$
$$3s\lambda + s\rho = -g^{\mu\nu} J_{\mu\nu}\,. \tag{D.16}$$

The combinations on the right side of these equations are computed explicitly as

$$\begin{aligned} g^{\mu\nu} J_{\mu\nu} &= \int dL\, Q'^2 = -2sB_0 \\ Q^\mu Q^\nu J_{\mu\nu} &= \int dL\, (Q \cdot Q')^2 = 2s^2 B_2 \,. \end{aligned} \tag{D.17}$$

Substituting these in Eq. (D.16) and solving for λ and ρ,

$$J_{\mu\nu} = (B_0 - B_2)\Big(P_\mu P_\nu - s g_{\mu\nu}\Big) + (3B_2 - B_0) Q_\mu Q_\nu \,, \tag{D.18}$$

which together with Eq. (D.7) yields the formula for $I^{(12)}_{\mu\nu}$ quoted in Eq. (4.25).

References

[1] J. F. Nieves and P. B. Pal, Phys. Rev. D **36**, 315 (1987).

[2] J. F. Nieves and P. B. Pal, Phys. Rev. D **64**, 076005 (2001)

[3] S. M. Bilenky, J. Hosek and S. T. Petcov, Phys. Lett. B **94**, 495 (1980).

[4] J. Schechter and J. W. F. Valle, Phys. Rev. D **22**, 2227 (1980).

[5] M. Doi, T. Kotani, H. Nishiura, K. Okuda and E. Takasugi, Phys. Lett. B **102**, 323 (1981).

[6] J. Schechter and J. W. F. Valle, Phys. Rev. D **23**, 1666 (1981).

[7] P. J. O'Donnell and U. Sarkar, Phys. Rev. D **52**, 1720 (1995)

[8] U. Sarkar and R. Vaidya, Phys. Lett. B **442**, 243 (1998)

[9] Y. Liu and U. Sarkar, Mod. Phys. Lett. A **16**, 603 (2001).

[10] S. M. Bilenky, S. Pascoli and S. T. Petcov, Phys. Rev. D **64**, 053010 (2001)

[11] J. A. Aguilar-Saavedra and G. C. Branco, Phys. Rev. D **62**, 096009 (2000)

[12] S. Pascoli, S. T. Petcov and L. Wolfenstein, Phys. Lett. B **524**, 319 (2002)

[13] S. T. Petcov, Sov. J. Nucl. Phys. **25**, 340 (1977); Errata **25**, 698 (1977). [Yad. Fiz. **25**, 641 (1977), Errata **25**, 1336 (1977)].

[14] See, e.g., *Quantum Field Theory*, C. Itzykson and J.B. Zuber, (McGraw-Hill, New York, 1980) pp. 315.

[15] O. W. Greenberg, Phys. Rev. D **32**, 1841 (1985).

[16] I. Dunietz, O. W. Greenberg and D. Wu, Phys. Rev. Lett. **55**, 2935 (1985).